CFT4 as SO(4,2)-invariant TFT2

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Based on "CFT4 as SO(4,2)-invariant TFT2"
R. de Mello Koch and S. Ramgoolam
arXiv:1403.6646 [hep-th], Nucl. Phys. B890
+ refs therein.

Background : Toy model for quantum field theory

- Toy Model : Gaussian integration

$$Z[J] = \int d\phi e^{-\frac{\phi^2}{2}+J\phi} = e^{-J^2/2}\sqrt{2\pi}$$

- Example of Correlators (or moments of distribution) is

$$< \phi\phi >= \frac{\int d\phi e^{-\frac{\phi^2}{2}}\phi^2}{\int d\phi e^{-\frac{\phi^2}{2}}} = 1$$

- General correlators can be computed by **Wick's theorem** : sum over pairings of which give a sum over $\prod < \phi\phi >$

$$< \phi^{2n} >= \text{ Sum over pairings } = 2^n n!$$

- Wick's theorem can be derived by relating $< \phi^{2n} >$ to the $2n$'th derivative of $Z[J]$.

Background : Simplest 4D quantum field theory

- Simplest Quantum field theory :

$$\phi \to \phi(x)$$

where $x \in \mathbb{R}^4$. The variable ϕ is now a real scalar field in four dimensions. QFT is the quantum dynamics of this scalar field.

- The integral is replaced by a path integral - an integral over the space of fields.

$$Z[J(x)] = \int D\phi(x) e^{-\int d^4x \partial_\mu \phi \partial_\mu \phi + J(x)\phi(x)}$$

- Correlators in this Gaussian field theory (or free field theory) are

$$< \phi(x_1)\phi(x_2)\cdots\phi(x_{2n}) > = \frac{\int D\phi e^{-\int d^4x \partial_\mu \phi \partial_\mu \phi} \phi(x_1)\cdots\phi(x_{2n})}{\int D\phi e^{-\int d^4x \partial_\mu \phi \partial_\mu \phi}}$$

Background : Simplest QFT

• The correlator is a function on $(\mathbb{R}^4)^{2n}$ depending on $2n$ space-time points.

• Basic two point function is now

$$< \phi(x_1)\phi(x_2) >= \frac{1}{(x_1 - x_2)^2}$$

• In QFT we are really interested in Minkowski space. The Lagrangian uses the Minkowski space metric

$$\int d^4 x \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$$

where $\eta = Diag(-1, 1, 1, 1)$.

• The correlators in the Minkowskian theory can be computed by Wick's rule again. The denominator in the 2-point function is $\eta^{\mu\nu}(x_1 - x_2)_\mu (x_1 - x_2)_\nu$.

Background : Simplest quantum field theory

• As in the case of ordinary Gaussian integration, we can get the general free field (Gaussian) correlators by using Wick's theorem

$$\langle\phi(x_1)\phi(x_2)\cdots\phi(x_{2n})\rangle = \sum_{pairings}\prod_{pairs}\frac{1}{(x_i - x_j)^2}$$

• Gaussian correlators give a zero-dimensional analog of 4D QFT.

• This simple QFT is an example of a **conformal field theory** - CFT. The symmetries of this theory include Lorentz invariance $SO(3,1)$; translational invariance with generators P_μ, hence Poincare invariance $ISO(3,1)$. But also scaling symmetry

$$x \to \lambda x$$
$$\phi \to \lambda^{-1}\phi$$

• The action of the theory is invariant under scaling

$$S = \int d^4x \partial_\mu\phi\partial^\mu\phi$$

• In general correlators involve insertions of **composite operators**

$$\phi(x_i) \rightarrow (\partial\partial..\partial\phi(x_i))(\partial\partial..\partial\phi(x_i))\cdots$$

• Can be done using Wick's theorem.

• Everything is much more non-trivial for interacting theory. e.g. Add $g \int d^4x\phi^4$ to the action.

• Perturbative QFT (expansion in small g) can be done ... involves renormalization, underlies quantum electrodynamics, the standard model of particle physics etc.

Conformal 4D Quantum field theory as TFT2

• The conformal symmetry is $SO(4,2)$ group. The Lie algebra $so(4,2)$ contains $so(4)$ generators $M_{\mu\nu}$, along with translations P_μ, scaling D and special conformal transformations K_μ.

• The main point of talk : This free field CFT4 (four-dimensional conformal quantum field theory) is an $so(4,2)$ invariant TFT2 (two-dimensional topological quantum field theory).

• The main suggestion of the talk : More general CFT4 - interacting as opposed to free - are also TFT2s.

• Evidence comes from

- relation between some distinguished correlators in the most interesting CFT4 to the simplest TFT2.
- Some recent results (of Frenkel-Libine) on $SO(4,2)$ equivariant interpretation of some conformal integrals of perturbative QFT.

OUTLINE

- TFT2 axiomatically defined in terms of
  - 2D cobordisms $\rightarrow$ Associative algebras, with non-degenerate pairing (Atiyah)
  - TFT2 with global symmetry (following Moore-Segal)
- Free field CFT4 two-point function from TFT2 perspective
  - An invariant state in a tensor product of $SO(4,2)$ representations
- $SO(4,2)$ invariant TFT2 for general CFT4 correlators
  - State space
  - Invariant Maps
  - Correlators.
  - Non-degeneracy and associativity.
- A most interesting interacting CFT4 : $N = 4$ SYM.
  – Simplest TFT2 from a sector of CFT4 correlators.
- Future directions and Open problems.

TFT2 - Axiomatic Approach

- Associate a vector space $\mathcal{H}$ to a circle - for explicit formulae choose basis e_A.

- Associate tensor products of $\mathcal{H}$ to disjoint unions.

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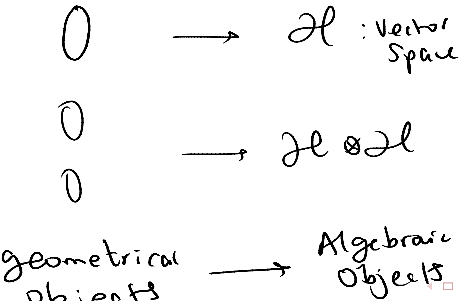
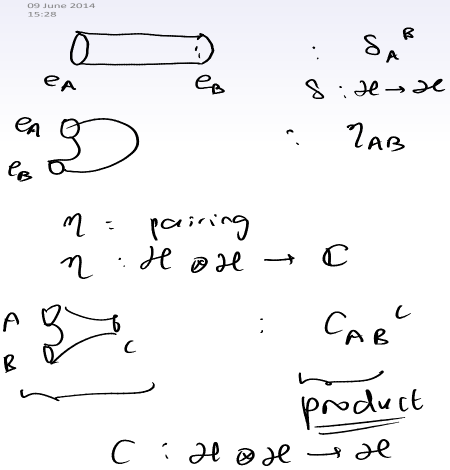


• Interpolating oriented surfaces between circles (cobordisms) are associated with linear maps between the vector spaces.



In math language, the circles are objects and interpolating surfaces (cobordisms) are morphisms in a geometrical category.

The vector spaces are objects, and linear maps are morphisms in an algebraic category.

The correspondence is a functor.

All relations in the geometrical side should be mirrored in the algebraic side.

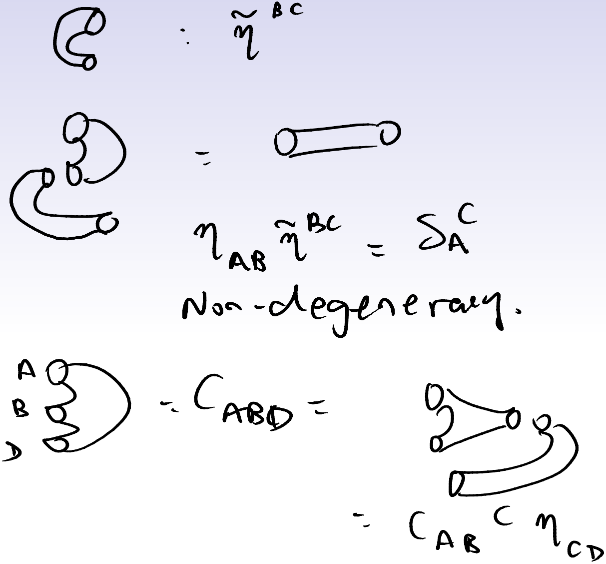
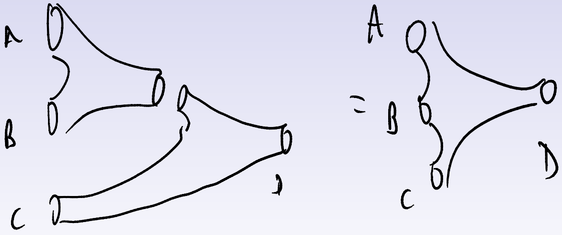
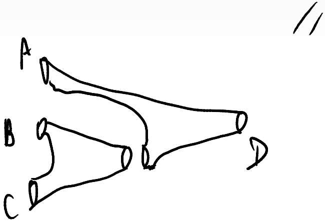


Figure: Non-degeneracy





2 gluings to give same cobordism

$$\Downarrow$$

$$C_{AB}^{E} C_{EC}^{D} = C_{BC}^{E} C_{EA}^{D}$$

Associativity

Figure: Associativity

To summarise, TFT2's correspond to commutative, associative, non-degenerate algebras - known as Frobenius algebras.

In the application at hand - the vector space is infinite dimensional. So we do not have a well-defined torus amplitude.

We take the usual Frobenius algebra equations, and consider a genus zero restriction. Mathematicians will probably prefer a more careful treatment of this point.

TFT2 with global symmetry group G

• The state space is a representation of a group G - which will be $SO(4,2)$ in our application.

• The linear maps are G-equvariant linear maps.

$$\begin{array}{ccc} V & \xrightarrow{\eta} & W \\ g \downarrow & & \downarrow g \\ V & \xrightarrow[\eta]{} & W \end{array}$$

$$g \circ \eta = \eta \circ g$$

Special case: $W = \mathbb{C}$

- trivial rep:
- invariant rep

$\Rightarrow \quad g \circ \eta = \eta$

The map η is invariant.

$\Rightarrow \quad \eta$ is an invt. map $V \otimes V \to \mathbb{C}$

Figure: G-equivariance

group: $g \cdot \eta(v_1, v_2) = \eta(gv_1, gv_2) = \eta(v_1, v_2)$

Lie algebra:

$$\mathcal{L}\,\eta(v_1, v_2) = \eta(\mathcal{L}v_1, v_2) + \eta(v_1, \mathcal{L}v_2)$$

$$= 0$$

$$\mathcal{L}_a{}^{a'}\eta_{a'b} + \mathcal{L}_b{}^{b'}\eta_{ab'} = 0$$

Example of η: $V \otimes V \to \mathbb{C}$; $V = V_{\frac{1}{2}}$ $SU(2)$

invariant: $\uparrow\downarrow - \downarrow\uparrow$

$\eta(\uparrow, \downarrow) = 1$; $\eta(\downarrow, \uparrow) = -1$

$\eta(\uparrow, \uparrow) = 0$; $\eta(\downarrow, \downarrow) = 0$

Figure: invariant map concrete example

PART II - Invariant linear maps and the basic CFT4 2-point function

- Free massless scalar field theory in four dimensions.

- The basic two-point function

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{(x_1 - x_2)^2}$$

- All correlators of composite operators are constructed from this using Wick contractions.

Ths theory has $SO(4,2)$ symmetry. Lie algebra spanned by $D, P_\mu, M_{\mu\nu}, K_\mu$ - Scaling operator, translations, $SO(4)$ rotations, and special conformal transformations.

• In radial quantization, there is a correspondence between local operators (composite operators) and quantum states we choose a point, say origin of Euclidean R^4, and we

$$
\begin{aligned}
&Lim_{x\to 0}\phi(x)|0>= v^+ \\
&Lim_{x\to 0}\partial_\mu\phi(x)|0>= P_\mu v^+ \\
&\vdots
\end{aligned}
$$

• The state v^+ is the lowest energy state, in a lowest-weight representation.

$$
\begin{aligned}
&Dv^+ = v^+ \\
&K_\mu v^+ = 0 \\
&M_{\mu\nu} v^+ = 0
\end{aligned}
$$

Higher energy states are generated by $S_l^{\mu_1\cdots\mu_l} P_{\mu_1}\cdots P_{\mu_l} v^+$, where S_l is a symmetric traceless tensor of $SO(4)$.

There is a dual representation V_-, which is a representation with negative scaling dimensions.

$$D v^- = -v^-$$
$$K_\mu v^- = 0$$
$$M_{\mu\nu} v^- = 0$$

Other states are generated by acting with $K \cdots K$.

There is an invariant map $\eta : V_+ \otimes V_- \to \mathbb{C}$.

No invariant in $V_+ \otimes V_+$ or $V_- \otimes V_-$.

$$\eta(v^+, v^-) = 1$$

The invariance condition determines η, e.g

$$\eta(P_\mu v^+, K_\nu v^-) = -\eta(v^+, P_\mu K_\nu v^-)$$
$$= \eta(v^+, (-2D\delta_{\mu\nu} + 2M_{\mu\nu})v^-) = 2\delta_{\mu\nu}$$

Using invariance conditions once finds that $\eta(P_\mu P_\mu v^+, v)$ is a null state. Setting this state to zero (imposing EOM), i.e defines a quotient of bigger representation V_+ which is the irreducible $\tilde{V}_+$. And makes η non-degenerate : no null vectors.

So we see that η is the kind of thing we need for TFT2 with $SO(4,2)$ symmetry. It has the non-degeneracy property and the invariance property.

Before relating this to the 2-point function, let us define a closely related quantity by taking the second field to the frame at infinity.

$$x_2' = \frac{x_2}{x_2^2}$$

$$\langle \phi(x_1)\phi'(x_2') \rangle = x_2^2 \langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{(1 - 2x_1 \cdot x_2' + x_1^2 x_2'^2)} \equiv F(x_1, x_2')$$

Now to link CFT4 to TFT2, calculate

$$\eta(e^{-iP \cdot x_1} v^+, e^{iK \cdot x_2'} v^-)$$

by using invariance and commutation relations as outlined above.

and find

$$\eta(e^{-iP \cdot x_1} v^+, e^{iK \cdot x_2'} v^-) = F(x_1, x_2')$$

So there is an invariant in $V_+ \otimes V_-$ and thus in $V_- \otimes V_+$, but not in $V_+ \otimes V_+$ or $V_- \otimes V_-$. It is useful to introduce $V = V_+ \oplus V_-$ and define $\eta : V \otimes V \to \mathbb{C}$.

$$\eta = \begin{pmatrix} 0 & \eta_{+-} \\ \eta_{-+} & 0 \end{pmatrix}$$

In V we have a state

$$\Phi(x) = \frac{1}{\sqrt{2}} \left(e^{-iP\cdot x} v^+ + x'^2 e^{iK\cdot x'} v^- \right)$$

so that

$$\eta(\Phi(x_1), \Phi(x_2)) = \frac{1}{(x_1 - x_2)^2}$$

This is the basic free field 2-point function, now constructed from the invariant map $\eta : V \otimes V \to \mathbb{C}$. The factor of 2 because $\eta(-,+)$ and $\eta(+,-)$ both contribute the same answer.

PART III : The TFT2 state space and amplitudes for CFT4 correlators

To get ALL correlators, we must set up a state space, which knows about composite operators.

• The states obtained by the standard operator state correspondence from general local operators are of the form

$$P_{\mu_1} \cdots P_{\mu_{k_1}} \phi \;\; P_{\mu_1} \cdots P_{\mu_{k_2}} \phi \;\; \cdots P_{\mu_1} \cdots P_{\mu_{k_n}} \phi$$

• Particular linear combinations of these are primary fields - irreducible representations (irreps) of $SO(4, 2)$.

• The list of primary fields in the n-field sector is obtained by decomposing into irreps the space

$$Sym(V_+^{\otimes n})$$

For the state space $\mathcal{H}$ of the TFT2 - which we associate to a circle in TFT2, we take

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} Sym(V^{\otimes n})$$

where $V = V_+ \oplus V_-$.

This state space is

- big enough to accommodate all the composite operators
- and admit an invariant pairing,
- small enough for the invariant pairing to be non-degenerate

Recall

$$\Phi(x) = \frac{1}{\sqrt{2}}(e^{-iP\cdot x}v^{+} + x'^{2}e^{iK\cdot x'}v^{-})$$

The state space contains

$$\Phi(x) \otimes \Phi(x) \otimes \cdots \Phi(x)$$

which is used to construct composite operators in the TFT2 set-up.

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(e^{-iP.x} v^{+} + (x')^{2} e^{iK.x'} v^{-} \right)$$

$$\begin{aligned}\Phi(x)\otimes\Phi(x) = \frac{1}{2}\Big\{ & e^{-iP.x} v^{+} \otimes e^{-iP.x} v^{+} \\ & + (x')^{2} e^{iK.x'} v^{-} \otimes (x')^{2} e^{iK.x'} v^{-} \\ & + e^{-iP.x} v^{+} \otimes (x')^{2} e^{iK.x'} v^{-} \\ & + (x')^{2} e^{iK.x'} v^{-} \otimes e^{-iP.x} v^{+} \Big\}\end{aligned}$$

$$\in \mathrm{Sym}(V^{\otimes 2})$$

Figure: Composite in symmetric product

• The pairing $\boldsymbol{\eta} : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ is constructed so as to be able to reproduce all the 2-point functions of arbitrary composite operators.

• The $\mathcal{H}$ is built from tensor products of V.

• The $\boldsymbol{\eta}$ is built from products of the elementary η, using Wick contraction sums.

$$V = V^{(0)} + V^{(1)} + V^{(2)} + V^{(3)} + \cdots$$

$$V^{(1)} \cong V, \quad V^{(2)} \cong \mathrm{Sym}(V^{\otimes 2}), \quad V^{(3)} \cong \mathrm{Sym}(V^{\otimes 3})$$

$$\eta(V^{(i)}, V^{(j)}) \propto \delta^{ij}$$

For $\eta(V^{(1)}, V^{(1)})$ use $\hat{\eta}$ defined before ;

$$\eta(V^{(2)}, V^{(2)}) :$$

$$= \eta(v_1 \otimes v_2, v_3 \otimes v_4)$$

$$= \hat{\eta}(v_1, v_3)\,\hat{\eta}(v_2, v_4) + \hat{\eta}(v_1, v_4)\,\hat{\eta}(v_2, v_3)$$

i.e. Apply Wick Contraction sums to generate products of the elementary $\hat{\eta}$.

Figure: Wick patterns for pairing

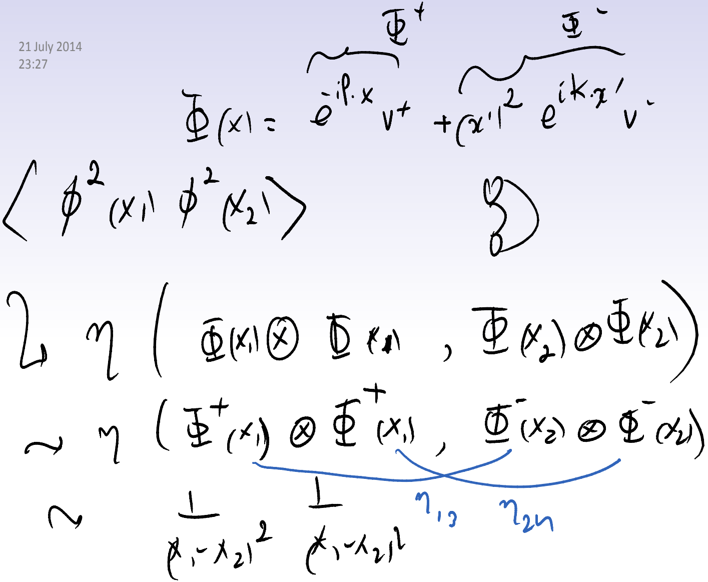


Figure: Computing correlators of CFT4 using TFT2 invariant maps

This defines the pairing η_{AB} where A, B take values in the space $\mathcal{H}$ - sum of all n-fold symmetric prducts of $V = V_+ \oplus V_-$.

The building blocks are invariant maps, so the product of these invariant maps is also invariant.

This is shown to be non-degenerate. Basically if you have a non-degenerate pairing $V \otimes V \to \mathbb{C}$, it extends to a non-degenerate pairing on $\mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$ – by using he sum over Wick patterns.

Hence

$$\eta_{AB}\tilde{\eta}^{BC} = \delta_A^C$$

The snake-cylinder equation.

Similarly can define 3-point functions

$$C_{ABC}$$

and higher

$$C_{ABC\ldots}$$

using Wick pattern products of the basic η's

By writing explicit formulae for these sums over Wick patterns, we can show that the associativity equations are satisfied.

• The C_{ABC} give 3-point functions. The $C_{AB}^{C} = C_{ABD}\tilde{\eta}^{DC}$ give OPE-coefficients. And the associativity equations of the TFT2 are the crossing equations of CFT4 - which are obtained by equating expressions for a 4-point correlator obtained by doing OPEs in two different ways.

• These properties of crossing symmetry and non-degeneracy are expected to be true for general CFTs - not just free CFTs, not just perturbative CFTs - can be argued based on properties of path integrals.

This is one reason for expecting that TFT2 should be relevant to CFT4 generally.

PART IV : Interesting interacting CFT4 and TFT2

• $N = 4$ SYM in 4D with $U(N)$ gauge group is dual to IIB superstring theory on $AdS_5 \times S^5$.

• Graviton and other Massless fields in 10D supergravity belong to one super-multiplet, which is half-BPS.
Annihilated by half the Q's (in addition to the S's)

• By KK reduction, gives rise to a tower of super-multiplets in 5D, labeled by an integer J.

• Of interest are extremal correlators of multi-trace operators - correspond to graviton interactions in the bulk, e.g.

$$\langle (\mathrm{tr}Z^{J_1}\mathrm{tr}Z^{J_2}\mathrm{tr}Z^{J_3}\cdots\mathrm{tr}Z^{J_k})(x_1)(\mathrm{tr}Z^{\dagger\ J'_1}\mathrm{tr}Z^{\dagger\ J'_2}\mathrm{tr}Z^{\dagger\ J'_3}\cdots\mathrm{tr}Z^{\dagger\ J'_k})(x_2)\rangle = \frac{1}{|x_1 - x_2|^{2n}} F(\vec{J}, \vec{J}')$$

• Fixing the dimension n of the holo operator, we have $J_1 + J_2 + \cdots J_k = n$, i.e. $\vec{J}$ form a partition of n.

• The function $F(\vec{J}, \vec{J}')$ has an elegant description in terms of 2-dimensional topological field theory (TFT2) based on the symmetric group S_n, of the $n!$ permutations of n objects.

• First hint : Partitions of n correspond to conjugacy classes of S_n.

• The TFT2 can be constructed from S_n 2D lattice gauge theory with a topological lattice action.

• The TFT2 can also be described in terms of 3-point functions C_{ABC} - (A, B , C label states in a vector space) which are building blocks for higher point functions by gluing.

• These C_{ABC} satisfy some associativity and non-degeneracy conditions which correspond to the properties of the geometrical gluing.

• This is the axiomatic approach of Atiyah - based on the category of cobordisms.

$$\langle O_{\sigma_1} \, O^{\dagger}_{\sigma_2} \rangle \longrightarrow \text{Correlator of 2 half BPS ops}$$

$$= \sum_{\substack{\sigma_1' \in T_1 \\ \sigma_2' \in T_2 \\ \sigma_3 \in S_n}} \delta(\sigma_1' \sigma_2' \sigma_3) \, N^{C_{\sigma_3}}$$

$$Z_{TFT2}(T_1, T_2, T_3)$$

$$= \sum_{\substack{\sigma_1' \in T_1 \\ \sigma_2' \in T_2 \\ \sigma_3' \in T_3}} \delta(\sigma_1' \sigma_2' \sigma_3')$$

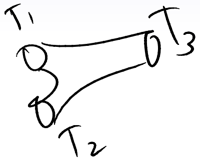


Figure: TFT2 structure from combinatorics of CFT4

PART V : Remarks, future directions, open problems.

• The $SO(4,2)$-invariant TFT2 has infinite dimensional state space. The genus zero restriction of the usual TFT2 equations makes sense. Restriction clear at level of equations - how to modify the axiomatics accordingly is an open problem.

• Simpler examples of TFT2 with infinite dimensional state spaces - can be constructed from 1-variable Gaussian integration ; or large k limit of $SU(2)$ WZW fusion rules.

• A **general** CFT has a non-degenerate inner product, and an operator product expansion - can be argued based on path integral representation.

$$\mathcal{O}_a(x)\mathcal{O}_b(0) = \sum_c x^{-2(\Delta_a/2+\Delta_b/2-\Delta_c/2)} C_{ab}^{\ c} \mathcal{O}_c(x)$$

• The operator product is believed to be associative for **general** CFT4. Based on path integral arguments. Hence ingredients for TFT2 in general CFT4. But given a set of correlators, consistent with associativity of the OPE not clear how to associate a sum involving positive dimension and negative dimension representations - which was needed to write correlators as $SO(4,2)$ equivariant maps.

• Look at perturbative CFT in this approach e.g $N = 4$ SYM.

• State space is the same as in free theory - but the invariant maps are not just Wick contractions, they involve the interaction vertices. Encouraging results from Frenkel-Libine who show that some conformal integrals from perturbation theory can be expressed in terms of $SO(4,2)$ equivariant maps.

• The problem of enumerating all the irreps in $Sym(V_+^{\otimes n})$ is surprisingly little understood.
Techniques of TFT2-techniques
using $SU(2) \times SU(2) \times$ 1-variable Polynomials
were useful in getting explicit answers for $n = 3$. We hope to go further and look at structure of these multiplicites for higher n.

Character of V_+ is

$$\sum_{n=0}^{\infty} s^{n+1} \chi_{n/2}(x) \chi_{n/2}(y)$$

The multplication rule for the $SU(2)$ characters is needed ...
hence the associative product on an algebra spanned by
X_j, Y_j, s
TO ADD :
1) some more formulae on matrix scalar
2) Explanations of why negative states are good for the correlators ..